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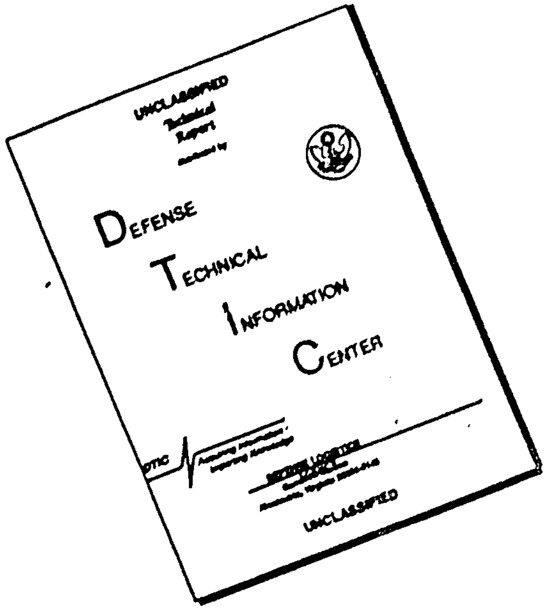

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TRANSLATION

EQUATION OF MOTION OF AN
AIRCRAFT WITH A JET ENGINE

By

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EQUATION OF MOTION OF AN AIRCRAFT WITH A JET ENGINE

By

G. M. Trakhtenberg

Introduction

Contemporary aircraft building is characterized by an even larger application of automation for flight control. In the method of established rectilinear flight, in the method of takeoff and landing the automatic pilot is widely used. Calculation of the characteristics of the automatic pilot is done on the basis of a contemporary solution of equations of the dynamics of the aircraft-automatic pilot control system.

For the collective solution of equations of the dynamics of an aircraft and autopilot equations are usually used from the motion of a "solid" aircraft in a mobile system of coordinates. However for aircraft, the mass of which greatly changes (as, for example, aircraft, supplied with liquid-fuel rocket engines, solid fuel jet engines), use of equations of a "solid" aircraft for analysis of the dynamics of a regulated system is a rough approximation.

Ignorance of the influence of variability of mass on the dynamics of the aircraft sometimes leads to errors in the engine arrangement on the aircraft. Variability of mass, as this is seen

from this article, can render significant influence on the damping characteristics of the aircraft. And besides, these characteristics depend on the location of the variable masses and the jet engine to the center of gravity of the aircraft.

Numerical calculations show that the damping influence of variable masses in reference to its own damping of the aircraft can reach separate sections of the flight trajectory up to 300%. Thus, for example, during take off of the aircraft with the help of an accelerator its own damping coefficient with respect to axis y_1 is equal to $m_y^{\omega_y} = -0.014$, and the additional damping coefficient from the accelerator and turbojet engine is equal to $\Delta m_y^{\omega_y} = -0.032$. Usually at the joint solution of equations of a closed aircraft-autopilot control system we do not consider additional damping from variability of masses. This circumstance can lead to errors in the selection of transfer coefficients of the automatic pilot at separate conditions of aircraft flight.

Listed in this article, the equations of aircraft motion are universal and applicable for aircraft of variable and constant mass.

Placement of the problem

Jet engines create large thrust at the expense of ejection of large masses of fuel mixture in a unit of time. Consequently, mass and moment of inertia of the aircraft significantly change in time. With these circumstances it is impossible to disregard in the equations of movement of the aircraft the variability of mass and moment of inertia.

For a conclusion of the equation of motion of the aircraft it would have been possible to use by equations of motion of a rocket

[1]. However, due to the peculiarities of the approach to the solution of the placed problem and in connection with the newly introduced conceptions, we will list the main hypotheses and computations of the conclusion of the equations of motion of the aircraft of variable mass.

The most general case of dynamics of an aircraft of variable mass is the motion of an aircraft with turbo-jet engines (TJE), inasmuch as side by side with the separation of gases from exhaust nozzles takes place continuous connection of air masses.

However, not losing community of proof, let us consider in the beginning the dynamics of a more simple case - an aircraft with a liquid-fuel rocket engine, but then we will spread the received result also in the case of an aircraft with a turbo-jet engine.

A liquid-fuel rocket engine works on a fuel mixture, which consists of a liquid fuel and a liquid oxidizer, located in special tanks, connected to the liquid-fuel rocket engine. Mass, moment of inertia and center of gravity of the aircraft change at the expense of consumption of the fuel mixture. In the process of flight the mixture proceeds in the engine and after combustion is ejected from the engine with great speed which also stipulates the appearance of reactive force. This reactive force is determined, practically, by the taking of experimental characteristics on a testing stand in an established method. However work of the engine on the aircraft together with the entire system of supplying of the fuel mixture is distinguished from the conditions, with which these characteristics are taken on the stand.

The engine with the whole system of supplying of the fuel mixture participates in all evolutions of the aircraft and its characteristics in dynamics, certainly, more general, than the characteristics, taken

isolated from the aircraft in statics.

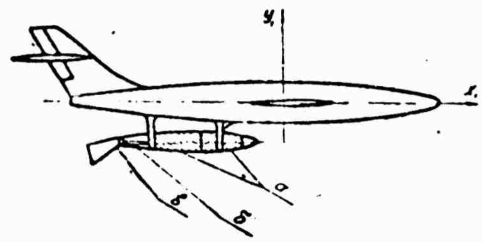

Fig. 1. Diagram of a liquid-fuel rocket engine: a) tanks with fuel mixture, b) wiring from tanks to engine, c) engine.

In figure 1 a liquid-fuel rocket engine is shown. If we revolve the aircraft around axis Y_1, then with this in the engine and system of supplying of the fuel mixture appear transverse force and moment due to angular velocity. Therefore, in addition to the characteristics of the engine, taken on the testing stand, we will consider the forces, appearing in the system of supplying of the fuel mixture of the aircraft during its rotation.

The main question which is the subject of our investigation, is the question of calculation of the dynamics of overflowing of the fuel mixture on the aircraft (including the engine) and the variability of its mass on the motion of the aircraft on the whole. For solution of this problem we will introduce some idealization, which would reflect the main problem, disregarding secondary insignificant phenomena.

The mass of the aircraft consists of two parts:

the first - constant and the second - variable (the fuel mixture), changing in flight with time (Fig. 1).

In reference to the constant mass the known motion equations are applicable in the system of coordinates connected with the aircraft.

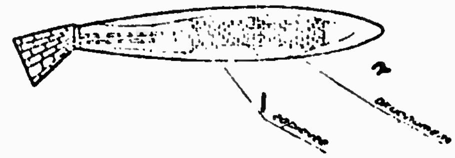

Fig. 2. Idealized diagram of a liquid-fuel rocket engine. 1) fuel; 2) oxidizer.

In relation to the second part (variable mass) it is necessary to preliminarily derive the equations of motion. In the conclusion of the equations of motion of the aircraft it is assumed that the law of motion of the fuel mixture in the combined system of coordinates is known or given beforehand.

Taking the fuel mixture as a "solid mass" into every moment of time, we will compose the following idealized calculated model of variable mass of an aircraft (Fig. 2). The variable mass (fuel mixture) consists of "solid bricks", moving from the tanks to the engine. In the engine every "brick", after the process of combustion, is separated from the aircraft with great speed, in consequence of which appears a reactive force. With respect to all remaining "bricks" of the variable mass of the aircraft are applicable, obviously, equations of the dynamics of a point, taking into account the circumstance that the "bricks" in the combined system of coordinates have relative velocities and accelerations.

Consequently, the constant mass of the aircraft can be considered also as variable, the elements of which have zero values of relative velocity. Therefore in the future we will not divide an aircraft into constant and variable parts, but will consider it on the whole as a variable mass.

1. Dynamics of an aircraft of variable mass with a liquid-fuel rocket engine

On the basis of the idealized model built by us we will consider the mass of an aircraft gradually decreasing, consisting of separate "solid" elements, having relative velocities and accelerations in a combined system of coordinates.

To the aircraft is applied a thrust, equal to the reactive force, determined experimentally on the testing stand.

Taking these positions as the base of our further reasonings we will drive equations of motion of an aircraft of variable mass. For a conclusion of general equations of motion of an aircraft we expediently use the combined system of coordinates, adopted in aerodynamics.

Let A be any fixed point of the aircraft, and B another point on the aircraft, moving relative to point A (Fig. 3).

We will take an immobile system of coordinates $Ox_g y_g z_\sigma$, and a system of axes, connected with the aircraft $Ax_1y_1z_1$. We will unite with straight points O, A and B. Having introduced OA, OB and AB corresponding to vectors ρ_a, p and r, we will write the following equality: $\overline{\rho} = \overline{\rho}_a + r$. Taking the geometric time derivative from vector $\overline{\rho}$, we will obtain the vector of speed of point B:

$$\overline{V} = \frac{d\overline{\rho}}{dt} = \frac{d\overline{\rho}_a}{dt} + \frac{dr}{dt}. \qquad (1)$$

Expressing vector $\overline{V}$ by a projection on the connected axes of coordinates, during rotation of the aircraft with angular velocity $\overline{\omega}$, we will obtain [3]:

$$V_{x_1} = V_{Ax_1} + \omega_{y_1} z_1 - \omega_{z_1} y_1 + \frac{dx_1}{dt},$$

$$V_{y_1} = V_{Ay_1} + \omega_{z_1} x_1 - \omega_{x_1} z_1 + \frac{dy_1}{dt},$$

$$V_{z_1} = V_{Az_1} + \omega_{x_1} y_1 - \omega_{y_1} x_1 + \frac{dz_1}{dt}, \qquad (2)$$

where $V_{Ax_1}, V_{Ay_1}, V_{Az_1}$ – are projections of absolute velocity of point A on the connected axes of the aircraft,

$\omega_{x_1}, \omega_{y_1}, \omega_{z_1}$ – are projections of angular velocity of the aircraft on the connected axes of the aircraft,

$\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}$ – are projections of relative velocity of point B on the connected axes of the aircraft.

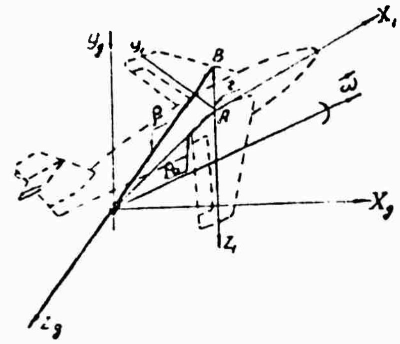

Fig. 3. Ground and connected systems of coordinates of an aircraft.

Having designed the vector of acceleration $\overline{w}$ on the connected axes of coordinates, we will find [3]:

$$w_{x_1} = \frac{dV_{Ax_1}}{dt} + \frac{d\omega_{y_1}}{dt} z_1 - \frac{d\omega_{z_1}}{dt} y_1 + 2\omega_{y_1}\frac{dz_1}{dt} - 2\omega_{z_1}\frac{dy_1}{dt} + \frac{d^2x_1}{dt^2} +$$
$$+ \omega_{y_1} V_{Az_1} + \omega_{y_1}\omega_{x_1} y_1 - \omega_{y_1}^2 x_1 - \omega_{z_1} V_{Ay_1} - \omega_{z_1}^2 x_1 + \omega_{z_1}\omega_{x_1} z_1,$$
$$w_{y_1} = \frac{dV_{Ay_1}}{dt} + \frac{d\omega_{z_1}}{dt} x_1 - \frac{d\omega_{x_1}}{dt} z_1 + 2\omega_{z_1}\frac{dx_1}{dt} - 2\omega_{x_1}\frac{dz_1}{dt} + \frac{d^2y_1}{dt^2} +$$
$$+ \omega_{z_1} V_{Ax_1} + \omega_{z_1}\omega_{y_1} z_1 - \omega_{z_1}^2 y_2 - \omega_{x_1} V_{Az_1} - \omega_{x_1}^2 y_1 + \omega_{x_1}\omega_{y_1} x_1,$$
$$w_{z_1} = \frac{dV_{Az_1}}{dt} + \frac{d\omega_{x_1}}{dt} y_1 - \frac{d\omega_{y_1}}{dt} x_1 + 2\omega_{x_1}\frac{dy_1}{dt} - 2\omega_{y_1}\frac{dx_1}{dt} + \frac{d^2z_1}{dt^2} +$$
$$+ \omega_{x_1} V_{Ay_1} + \omega_{x_1}\omega_{z_1} x_1 - \omega_{x_1}^2 z_1 - \omega_{y_1} V_{Ax_1} - \omega_{y_1}^2 z_1 + \omega_{y_1}\omega_{z_1} y_1. \qquad (3)$$

We will use for composition of equations of dynamics of an aircraft expressions (3) for speeds and accelerations of elementary points of the aircraft in projections on connected axes of coordinates.

We will write equations of forces of the dynamics of the system:

$$\sum_{i=1}^{n} m_i w_{x_i i} = \sum_{i=1}^{n} F_{x_i i} \quad (а), \quad \sum_{i=1}^{n} m_i w_{y_i} = \sum_{i=1}^{n} F_{y_i} \quad (б), \quad \sum_{i=1}^{n} m_i w_{z_i} = \sum_{i=1}^{n} F_{z_i} \quad (в), \qquad (4)$$

where $\sum_{i=1}^{n} F_{x_i i}$, $\sum_{i=1}^{n} F_{y_i i}$, $\sum_{i=1}^{n} F_{z_i i}$ — are projections of the vector of external forces, applied to the system, n is the quantity of elementary particles of the aircraft in a given moment of time.

We will preliminarily introduce the following conceptions, having a real physical meaning, which we will use in the future.

1. Speed of change of the center of gravity of the aircraft at constant mass.

2. Speed of change of the center of gravity of the aircraft at variable mass.

3. Speed of change of the moment of inertia of the aircraft at constant mass.

4. Speed of change of the moment of inertia of the aircraft at variable mass.

We will explain this examples:

a) let us have on the aircraft two tanks with a combustible liquid. One tank - in the nose, and the other - in the tail, and let the liquid from the nose tank overflow into the tail tank. The position of the center of gravity of the aircraft (x_{cg}, y_{cg}) changes and is moved to the tail. The Mass of the aircraft with this does not change. In this case we distinguish the speed of the center of gravity of the aircraft at constant mass and designate correspondingly:

$$\frac{dx_{ц\,т}}{dt}(m = \text{const}),\ \frac{dy_{ц\,т}}{dt}(m = \text{const}).$$

The moments of inertia of the aircraft with respect to axes x_1, y_1, z_1 with this also changes and the speeds of their change will be recorded:

$$\frac{dI_{x_1}}{dt}(m = \text{const}),\ \frac{dI_{y_1}}{dt}(m = \text{const}),\ \frac{dI_{z_1}}{dt}(m = \text{const});$$

b) let the combustible liquid continuously overflow from the nose tank in tail tank and then flow from the aircraft with a determined speed. In this case the center of gravity of the aircraft is also moved, but its mass with this is variable. In this case we distinguish the speed of the center of gravity of the aircraft at variable mass and designate as:

$$\frac{dx_{ц\,т}}{dt}(m = \text{var}),\ \frac{dy_{ц\,т}}{dt}(m = \text{var}).$$

The moments of inertia with respect to axes x_1, y_1, z_1, changing not only due to the change in coordinates, but also because of the change of the mass and speed if their change, we designate correspondingly:

$$\frac{dI_{x_1}}{dt}(m = \text{var}),\ \frac{dI_{y_1}}{dt}(m = \text{var}),\ \frac{dI_{z_1}}{dt}(m = \text{var}).$$

In the future these physically clear conceptions we will also use.

For brevity of spelling of derivatives we will introduce the following conditional designations:

$$\frac{dx_{\text{ц т}}}{dt}(m=\text{const}) \to \left(\frac{dx_{\text{ц т}}}{dt}\right)_{mc},$$
$$\frac{dx_{\text{ц т}}}{dt}(m=\text{var}) \to \left(\frac{dx_{\text{ц т}}}{dt}\right)_{mv},$$
$$\frac{dI_{x_1}}{dt}(m=\text{const}) \to \left(\frac{dI_{x_1}}{dt}\right)_{mc} \text{ и т. д.}$$

Having substituted into equation (4a) the expression for w_{x_1}, we obtain:

$$\begin{aligned}&\frac{dV_{Ax_1}}{dt}\sum_{i=1}^{n}m_i + \frac{d\omega_{y_1}}{dt}\sum_{i=1}^{n}m_i z_{1i} - \frac{d\omega_{z_1}}{dt}\sum_{i=1}^{n}m_i y_{1i} + 2\omega_{y_1}\sum_{i=1}^{n}m_i\frac{dz_{1i}}{dt} - \\ &- 2\omega_{z_1}\sum_{i=1}^{n}m_i\frac{dy_{1i}}{dt} + \sum_{i=1}^{n}m_i\frac{d^2x_1}{dt^2} + \omega_{y_1}V_{Az_1}\sum_{i=1}^{n}m_i + \omega_{y_1}\omega_{x_1}\sum_{i=1}^{n}m_i y_{1i} - \\ &- \omega_{y_1}^2\sum_{i=1}^{n}m_i x_{1i} - \omega_{z_1}V_{Ay_1}\sum_{i=1}^{n}m_i - \omega_{z_1}^2\sum_{i=1}^{n}m_i x_{1i} + \omega_{z_1}\omega_{x_1}\sum_{i=1}^{n}m_i z_{1i} = \sum_{i=1}^{n}F_{x_1 i}.\end{aligned} \tag{5}$$

The obtained equation (5) can be simplified, if one considers the following dependences:

1. The sum of elementary masses of the aircraft is equal to the mass of the entire aircraft:

$$\sum_{i=1}^{n}m_i = M. \tag{6a}$$

2. The sum of the static moments of the elementary masses of the aircraft is equal to the static moment of its whole mass:

$$\sum_{i=1}^{n}m_i z_{1i} = Mz_{\text{ц т}},\ \sum_{i=1}^{n}m_i y_{1i} = My_{\text{ц т}},\ \sum_{i=1}^{n}m_i x_{1i} = Mx_{\text{ц т}}. \tag{6b}$$

3. The sum of the products of the elementary masses of the aircraft on corresponding relative velocities is equal to the product of the entire mass on the speed of the center of gravity of the aircraft at its constancy:

$$\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} = \frac{d}{dt} \sum_{i=1}^{n} m_i y_{1i} = M \left(\frac{dy_{\text{ц т}}}{dt} \right)_{mc},$$
$$\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} = \frac{d}{dt} \sum_{i=1}^{n} m_i x_{1i} = M \left(\frac{dx_{\text{ц т}}}{dt} \right)_{mc},$$
$$\sum_{i=1}^{n} m_i \frac{dz_{1i}}{dt} = \frac{d}{dt} \sum_{i=1}^{n} m_i z_i = M \left(\frac{dz_{\text{ц т}}}{dt} \right)_{mc}. \tag{6c}$$

4. The sum of the products of the elementary masses on the corresponding relative accelerations is equal to the mass of the entire aircraft on the average value of relative acceleration:

$$\sum_{i=1}^{n} m_i \frac{d^2 x_{1i}}{dt^2} = M \frac{d^2 x_{\text{cp}}}{dt^2}, \quad \sum_{i=1}^{n} m_i \frac{d^2 y_{1i}}{dt^2} = M \frac{d^2 y_{\text{cp}}}{dt^2}, \quad \sum_{i=1}^{n} m_i \frac{d^2 z_{1i}}{dt^2} = M \frac{d^2 z_{\text{cp}}}{dt^2}, \tag{6d}$$

where $\frac{d^2 x_{\text{cp}}}{dt^2}, \frac{d^2 y_{\text{cp}}}{dt^2}, \frac{d^2 z_{\text{cp}}}{dt^2}$ -- are the average values of projections of relative acceleration of the variable mass of the aircraft on the connected axes of coordinates.

Let us note that at a constant speed of supplying of liquid in the engine the average values of projections of relative accelerations are equal to zero. Having substituted dependences (6a), (6b), (6c), (6d) into equation (5), we obtain:

$$M \left[\frac{dV_{Ax_1}}{dt} + \frac{d\omega_{y_1}}{dt} z_{\text{ц т}} - \frac{d\omega_{z_1}}{dt} y_{\text{ц т}} + 2\omega_{y_1} \left(\frac{dz_{1\,\text{ц т}}}{dt} \right)_{mc} - 2\omega_{z_1} \left(\frac{dy_{1\,\text{ц т}}}{dt} \right)_{mc} + \frac{d^2 x_{\text{cp}}}{dt^2} + \omega_{y_1} V_{Az_1} + \omega_{y_1} \omega_{x_1} y_{\text{ц т}} - \omega_{y_1}^2 x_{\text{ц т}} - \omega_{z_1} V_{Ay_1} - \omega_{z_1}^2 x_{\text{ц т}} + \omega_{z_1} \omega_{x_1} z_{\text{ц т}} \right] = \sum_{i=1}^{n} F_{x_{1i}}. \tag{7}$$

Analogous to (4a), equations (4b) and (4c) take the following form:

$$M\left[\frac{dV_{Ay_1}}{dt}+\frac{d\omega_{z_1}}{dt}x_{\text{ц т}}-\frac{d\omega_{x_1}}{dt}z_{\text{ц т}}+2\omega_{z_1}\left(\frac{dx_{1\,\text{ц т}}}{dt}\right)_{mc}-2\omega_{x_1}\left(\frac{dz_{\text{ц т}}}{dt}\right)_{mc}+\right.$$
$$\left.+\frac{d^2y_{cp}}{dt^2}+\omega_{z_1}V_{Ax_1}+\omega_{z_1}\omega_{y_1}z_{\text{ц т}}-\omega_{z_1}^2y_{\text{ц т}}-\omega_{x_1}V_{Az_1}-\omega_{x_1}^2y_{\text{ц т}}+\omega_{x_1}\omega_{y_1}x_{\text{ц т}}\right]=$$
$$=\sum_{i=1}^{n}F_{y_{1i}},\quad M\left[\frac{dV_{Az_1}}{dt}+\frac{d\omega_{x_1}}{dt}y_{\text{ц т}}-\frac{d\omega_{y_1}}{dt}x_{\text{ц т}}+2\omega_{x_1}\left(\frac{dy_{1\,\text{ц т}}}{dt}\right)_{mc}-\right. \tag{8}$$
$$-2\omega_{y_1}\left(\frac{dx_{1\,\text{ц т}}}{dt}\right)_{mc}+\frac{d^2z_{cp}}{dt^2}+\omega_{x_1}V_{Ay_1}+\omega_{x_1}\omega_{z_1}x_{\text{ц т}}-\omega_{x_1}^2z_{\text{ц т}}-\omega_{y_1}V_{Ax_1}-$$
$$\left.-\omega_{y_1}^2z_{\text{ц т}}+\omega_{y_1}\omega_{z_1}y_{\text{ц т}}\right]=\sum_{i=1}^{n}F_{z_{1i}},$$

We obtained equations of dynamic equilibrium of the variable mass of aircraft in projections of forces on axes x_1, y_1, z_1. However these equations are insufficient for description of the conditions of full dynamic equilibrium of the system. To it is necessary still to add equations of moments around axes x_1, y_1, z_1:

$$-\sum_{i=1}^{n}m_i(w_{z_1}y_1-w_{y_1}z_1)_i+\sum_{i=1}^{n}M_{x_1}(F_i)=0, \tag{9a}$$
$$-\sum_{i=1}^{n}m_i(w_{x_1}z_1-w_{z_1}x_1)_i+\sum_{i=1}^{n}M_{y_1}(F_i)=0, \tag{9b}$$
$$-\sum_{i=1}^{n}m_i(w_{y_1}x_1-w_{x_1}y_1)_i+\sum_{i=1}^{n}M_{z_1}(F_i)=0. \tag{9c}$$

Having substituted into equations (9) the expressions for accelerations w_1, w_1, w_1 and considering with this that the mass will be displaced symmetrically relative to the longitudinal plane of the aircraft and that the speed of its change is constant

$$\left(z_{\text{ц т}}=I_{x_1z_1}=I_{y_1z_1}=\sum_{i=1}^{n}m_i\frac{dx_{1i}}{dt}z_{1i}=\sum_{i=1}^{n}m_i\frac{d^2y_{1i}}{dt^2}z_{1i}=\sum_{i=1}^{n}m_i\frac{d^2z_{1i}}{dt^2}y_{1i}=\right.$$
$$\left.=\frac{d^2z_{cp}}{dt^2}=\frac{d^2y_{cp}}{dt^2}=\frac{d^2x_{cp}}{dt^2}=0\right),$$

we obtain:

$$\left(\frac{dV_{Az_1}}{dt} + \omega_{x_1} V_{Ay_1} - \omega_{y_1} V_{Ax_1}\right) My_{\text{ц т}} + \frac{d\omega_{x_1}}{dt} I_{x_1} + 2\omega_{x_1}\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} y_{1i} +$$
$$+ 2\omega_{x_1}\sum_{i=1}^{n} m_i \frac{dz_{1i}}{dt} z_{1i} + \left(\omega_{x_1}\omega_{z_1} - \frac{d\omega_{y_1}}{dt}\right) I_{x_1y_1} + \omega_{y_1}\omega_{z_1}(I_{z_1} - I_{y_1}) -$$
$$- 2\omega_{y_1}\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} y_{1i} = \sum_{i=1}^{n} M_{x_1}(F_i). \tag{10a}$$

$$-\left(\frac{dV_{Az_1}}{dt} + \omega_{x_1} V_{Ay_1} - \omega_{y_1} V_{Ax_1}\right) Mx_{\text{ц т}} + \frac{d\omega_{y_1}}{dt} I_{y_1} + 2\omega_{y_1}\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} x_{1i} +$$
$$+ 2\omega_{y_1}\sum_{i=1}^{n} m_i \frac{dz_{1i}}{dt} z_{1i} - \left(\omega_{y_1}\omega_{z_1} + \frac{d\omega_{x_1}}{dt}\right) I_{x_1y_1} + \omega_{z_1}\omega_{x_1}(I_{x_1} - I_{z_1}) -$$
$$- 2\omega_{x_1}\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} x_{1i} = \sum_{i=1}^{n} M_{y_1}(F_i). \tag{10b}$$

$$\left(\frac{dV_{Ay_1}}{dt} + \omega_{z_1} V_{Ax_1} - \omega_{x_1} V_{Az_1}\right) Mx_{\text{ц т}} + \frac{d\omega_{z_1}}{dt} I_{z_1} + 2\omega_{z_1}\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} x_{1i} + \tag{10c}$$
$$+ (\omega_{y_1}^2 - \omega_{x_1}^2) I_{x_1y_1} + \omega_{x_1}\omega_{y_1}(I_{y_1} - I_{x_1}) - \left(\frac{dV_{Ax_1}}{dt} + \omega_{y_1} V_{Az_1} - \omega_{z_1} V_{Ay_1}\right) My_{\text{ц т}} +$$
$$+ 2\omega_{z_1}\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} y_{1i} = \sum_{i=1}^{n} M_{z_1}(F_i).$$

Till now in the conclusion of the equations we did not place limitations on the selection point of the origin of the coordinates of the connected system. In the selection of the origin of coordinates in the immediate center of gravity of the aircraft the static moments are equal to zero and, consequently, the members of the preceding equations (7), (8), (10), containing static moments relative to the center of gravity, vanish.

In the preceding equations (10) enter also members, containing sums:

$$\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} y_{1i}, \quad \sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} x_{1i}.$$

These sums can be interpreted, correspondingly, as derivatives of the product of inertia according to time at a separate change in coordinates x_1, y_1. Having differentiated the product of inertia of the entire aircraft according to time at constant mass, we obtain:

$$\frac{dI_{x_1y_1}}{dt}=\sum_{i=1}^{n} m_i \frac{d(x_1y_1)_i}{dt}=\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} y_{1i}+\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} x_{1i}. \qquad (11)$$

We introduce the following designations:

$$\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} y_{1i}=\frac{dI_{x_1y_1}(x)}{dt}mc, \quad \sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} x_{1i}=\frac{dI_{x_1y_1}(y)}{dt}mc. \qquad (12)$$

If we select the origin of the connected system of coordinates in the center of gravity, considering dependences (12) and (13)

$$\begin{aligned}
&\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt}=\frac{d}{dt}\sum_{i=1}^{n} m_i y_{1i}=m\frac{dy_{1T}}{dt} \ (m=\text{const}),\\
&\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt}=\frac{d}{dt}\sum_{i=1}^{n} m_i x_{1i}=m\frac{dx_{1T}}{dt} \ (m=\text{const}),\\
&\sum_{i=1}^{n} m_i \frac{dx_{1i}}{dt} x_{1i}=\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n} m_i x_{1i}^2 \ (m=\text{const}),\\
&\sum_{i=1}^{n} m_i \frac{dy_{1i}}{dt} y_{1i}=\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n} m_i y_{1i}^2 \ (m=\text{const}),\\
&\sum_{i=1}^{n} m_i \frac{dz_{1i}}{dt} z_{1i}=\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n} m_i z_{1i}^2 \ (m=\text{const}),\\
&\left(\frac{dI_{z_1}}{dt}\right)_{mc}=\frac{d}{dt}\sum_{i=1}^{n} m_i(x_{1i}^2+y_{1i}^2) \ (m=\text{const}),\\
&\left(\frac{dI_{x_1}}{dt}\right)_{mc}=\frac{d}{dt}\sum_{i=1}^{n} m_i(y_{1i}^2+z_{1i}^2) \ (m=\text{const}),\\
&\left(\frac{dI_{y_1}}{dt}\right)_{mc}=\frac{d}{dt}\sum_{i=1}^{n} m_i(x_{1i}^2+z_{1i}^2)_{mc}.
\end{aligned} \qquad (13)$$

then the equations of motion of the aircraft (7), (8), (10) take the following form:

$$
\begin{aligned}
&M\left[\frac{dV_{x_1}}{dt}+\omega_{y_1}V_{z_1}-\omega_{z_1}V_{y_1}-2\omega_{z_1}\left(\frac{dy_{\text{ц т}}}{dt}\right)_{mc}\right]=\sum_{i=1}^{n}F_{xi},\\
&M\left[\frac{dV_{y_1}}{dt}+\omega_{z_1}V_{x_1}-\omega_{x_1}V_{z_1}+2\omega_{z_1}\left(\frac{dx_{\text{ц т}}}{dt}\right)_{mc}\right]=\sum_{i=1}^{n}F_{yi},\\
&M\left[\frac{dV_{z_1}}{dt}+\omega_{x_1}V_{y_1}-\omega_{y_1}V_{x_1}+2\omega_{x_1}\left(\frac{dy_{\text{ц т}}}{dt}\right)_{mc}-2\omega_{y_1}\left(\frac{dx_{\text{ц т}}}{dt}\right)_{mc}\right]=\sum_{i=1}^{n}F_{zi},\\
&I_{x_1}\frac{d\omega_{x_1}}{dt}+\omega_{x_1}\left(\frac{dI_{x_1}}{dt}\right)_{mc}+\left(\omega_{x_1}\omega_{z_1}-\frac{d\omega_{y_1}}{dt}\right)I_{x_1y_1}+\omega_{y_1}\omega_{z_1}(I_{z_1}-I_{y_1})-\\
&\qquad-2\omega_{y_1}\left(\frac{dI_{x_1y_1}}{dt}(y)\right)_{mc}=\sum_{i=1}^{n}M_{x_1}(F_i),\\
&I_{y_1}\frac{d\omega_{y_1}}{dt}+\omega_{y_1}\left(\frac{dI_{y_1}}{dt}\right)_{mc}-\left(\omega_{y_1}\omega_{z_1}+\frac{d\omega_{x_1}}{dt}\right)I_{x_1y_1}+\omega_{z_1}\omega_{x_1}(I_{x_1}-I_{z_1})-\\
&\qquad-2\omega_{x_1}\left(\frac{dI_{x_1y_1}}{dt}(y)\right)_{mc}=\sum_{i=1}^{n}M_{y_1}(F_i),\\
&I_{z_1}\frac{d\omega_{z_1}}{dt}+\omega_{z_1}\left(\frac{dI_{z_1}}{dt}\right)_{mc}+(\omega_{y_1}^2-\omega_{x_1}^2)I_{x_1y_1}+\omega_{x_1}\omega_{y_1}(I_{y_1}-I_{x_1})=\sum_{i=1}^{n}M_{z_1}(F_i).
\end{aligned}
\tag{14}
$$

In the right sides of the obtained equations are contained only external forces. External forces and moments, acting on the aircraft, are:

1) aerodynamic forces and moments X_1, Y_1, Z_1, M_{x_1}, M_{y_1}, M_{z_1},

2) gravity G,

3) reactive force (thrust) Φ and moment from thrust M_{φ}.

Aerodynamic moments and forces at linear approximation are

expressed by the following dependences:

$$
\begin{aligned}
M_{x_1} &= (m_{x_1}^{\omega_{x_1}} \omega_{x_1} + m_{x_1}^{\omega_{y_1}} \omega_{y_1} + m_{x_1}^{\delta_э} \delta_э + m_{x_1}^{\delta_н} \delta_н + m_{x_1}^{\beta} \beta) Sql, \\
M_{y_1} &= (m_{y_1}^{\omega_{y_1}} \omega_{y_1} + m_{y_1}^{\omega_{x_1}} \omega_{x_1} + m_{y_1}^{\delta_э} \delta_э + m_{y_1}^{\delta_н} \delta_н + m_{y_1}^{\beta} \beta) Sql, \\
M_{z_1} &= (m_{z_0} + m_z^{\alpha} \alpha + m_z^{\dot\alpha} \dot\alpha + m_{z_1}^{\omega_z} \omega_z + m_z^{\delta_в} \delta_в) Sqb_a. \\
X_1 &= c_{x_1} Sq, \quad Y_1 = c_{y_1} Sq, \quad Z_1 = c_{z_1} Sq.
\end{aligned}
\tag{15}
$$

These coefficients $m_{y_1}^{\omega y}$, $m_{x_1}^{\omega x}$, c_y, c_x... depend on M numbers, angles of attack, and also on the position of the center of gravity of the aircraft. Since the equations of the motion of the aircraft are composed relative to the system of coordinates, passing through the center of gravity, the position of which changes in time, then it is natural that also the aerodynamic coefficients are re-counted relative to the current position of the center of gravity.

Having substituted equality (15) into equations (14), we obtain

$$
\begin{aligned}
&M\left[\frac{dV_{x_1}}{dt} + \omega_{y_1} V_{z_1} - \omega_{z_1}\left(V_{y_1} + 2\frac{dy_{\text{цт}}}{dt} mc\right)\right] = \Phi_{x_1} + c_{x_1} Sq - G \sin\theta, \\
&M\left[\frac{dV_{y_1}}{dt} + \omega_{z_1}\left(V_{x_1} + 2\frac{dx_{\text{цт}}}{dt} mc\right) - \omega_{x_1} V_{z_1}\right] = \Phi_{y_1} + c_{y_1} Sq - G \cos\gamma \cos\theta, \\
&M\left[\frac{dV_{z_1}}{dt} + \omega_{x_1}\left(V_{y_1} + 2\frac{dy_{\text{цт}}}{dt} mc\right) - \omega_{y_1}\left(V_{x_1} + 2\frac{dx_{\text{цт}}}{dt} mc\right)\right] = \\
&\qquad = c_{z_1} Sq + G \cos\theta \sin\gamma, \\
&I_{x_1} \frac{d\omega_{x_1}}{dt} + \left(\omega_{x_1}\omega_{z_1} - \frac{d\omega_{y_1}}{dt}\right) I_{x_1 y_1} + \omega_{y_1}\omega_{z_1}(I_{z_1} - I_{y_1}) = [m_{x_1}^{\delta_э}\delta_э + m_{x_1}^{\delta_н}\delta_н + \\
&\quad + m_{x_1}^{\beta}\beta + \left(m_{x_1}^{\omega_{x_1}} - \frac{dI_{x_1}}{dt} mc \frac{1}{Sql}\right)\omega_{x_1} + \left(m_{x_1}^{\omega_{y_1}} + 2\frac{dI_{x_1 y_1}(x)}{dt} mc \frac{1}{Sql}\right)\omega_{y_1}\Big] Sql, \\
&I_{y_1} \frac{d\omega_{y_1}}{dt} - \left(\omega_{y_1}\omega_{z_1} + \frac{d\omega_{x_1}}{dt}\right) I_{x_1 y_1} + \omega_{z_1}\omega_{x_1}(I_{x_1} - I_{z_1}) = \Big[m_{y_1}^{\delta_э}\delta_э + m_{y_1}^{\delta_н}\delta_н + \\
&\quad + m_{y_1}^{\beta}\beta + \left(m_{y_1}^{\omega_{y_1}} - \frac{dI_{y_1}}{dt} mc \frac{1}{Sql}\right)\omega_{y_1} + \left(m_{y_1}^{\omega_{x_1}} + 2\frac{dI_{x_1 y_1}(y)}{dt} mc \frac{1}{Sql}\right)\omega_{x_1}\Big] Sql, \\
&I_{z_1} \frac{d\omega_{z_1}}{dt} + (\omega_{y_1}^2 - \omega_{x_1}^2) I_{x_1 y_1} + \omega_{x_1}\omega_{y_1}(I_{y_1} - I_{x_1}) = \Big[m_{z_0} + m_z^{\alpha}\alpha + m_z^{\dot\alpha}\dot\alpha + \\
&\quad + m_z^{\delta_в}\delta_в + \left(m_{z_1}^{\omega_{z_1}} - \frac{dI_{z_1}}{dt} mc \frac{1}{Sqb_a}\right)\omega_{z_1}\Big] Sqb_a + M_{z_1}(\Phi.
\end{aligned}
\tag{16}
$$

The obtained equations (6) describe the motion of the aircraft in the general case of curvilinear flight. The motion of the aircraft is interconnected in all three planes.

During longitudinal motion ω_{y_1}, $\omega_{x_1} = V_{z_1} = y = Z_1 = 0$ general equations of motion (16) degenerate into the equations of longitudinal motion of the aircraft:

$$
\begin{aligned}
&M\left[\frac{dV}{dt} - 2\omega_{z_1}\left(\frac{dy_{\text{п т}}}{dt} mc\cos\alpha_\text{т} + \frac{dx_{\text{п т}}}{dt} mc\sin\alpha_\text{т}\right)\right] = \\
&\quad = \Phi\cos(\alpha_\text{т} + \varphi) - C_{x_1}\frac{S\rho V^2}{2} - G\sin\theta, \\
&MV\left[\frac{d\theta}{dt} + \frac{2\omega_z}{V}\left(\frac{dx_{\text{п т}}}{dt} mc\cos\alpha_\text{т} - \frac{dy_{\text{п т}}}{dt} mc\sin\alpha_\text{т}\right)\right] = \\
&\quad = \Phi\sin(\alpha_\text{т} + \varphi) + C_{y_1}\frac{S\rho V^2}{2} - G\cos\theta, \\
&I_{z_1}\frac{d\omega_z}{dt} = \left[\left(m_{z_1}^{\omega_{z_1}} - \frac{dI_{z_1}}{dt} mc\frac{1}{Sqb_a}\right)\omega_{z_1} + m_{z_0} + m_z^\alpha\alpha + m_z^{\dot\alpha}\dot\alpha + \right. \\
&\quad \left. + m_z^{\delta_\text{в}}\delta_\text{в}\right] Sqb_a + \Phi y_\Phi\cos\varphi,
\end{aligned}
\tag{17}
$$

where y_Φ is the arm of reactive force relative to the center of gravity, φ is the angle between the direction of reactive force and the axis of the aircraft (Fig. 4).

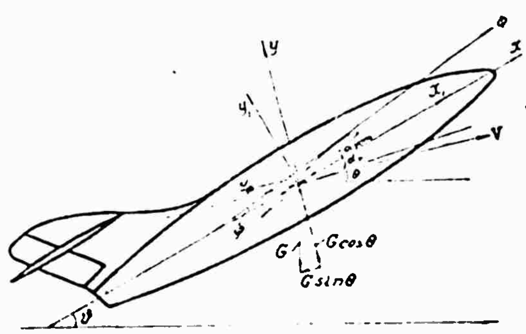

Fig. 4. Diagram of the forces, applied to an aircraft in the longitudinal plane.

If one takes in the obtained equations (16), (17) the relative velocity of fuel mixture of the aircraft equal to zero, then they identically coincide with the equations the "solid of" aircraft. However it is necessary to underline that although the obtained equations remind us of equations of motion of "solid" aircraft, they are considerably distinguished, and namely: the equations of an aircraft with a liquid-fuel rocket engine contain mass and moments of inertia relative to the connected axes - variable in time, the origin of coordinates of which coincides with the instantaneous center of gravity of the aircraft in flight, and also additional members, considering the relative motion of the fuel mixture in the aircraft.

In conclusion we will spread the obtained equations of dynamics of an aircraft with a liquid-fuel rocket engine in the case of a turbo-jet engine.

Analyzing consecutively our computations, it is easy to see that we nowhere used the peculiarities of the variable mass, concluded in the separation of combustible gases or in the connection of air masses. It is important that the variable mass has an influence on the dynamics of the aircraft - this is the presence of relative velocity of particles of the fuel mixture in the connected system of coordinates. Consequently, the obtained equations are completely spread in the case of a turbojet engine and are equations of an aircraft of variable mass in the most general form. However, at determination of derivatives of the moments of inertia, the coordinates of the center of gravity of the aircraft, entering into the equations of motion, one should consider the peculiarity of that or any other jet engine, i.e. the separated and joined masses of the aircraft. Below are listed formulas for determination of the corresponding derivatives

at constant mass (Para. 2).

Analyzing the obtained equations (16), (17), it is possible to come to following qualitative conclusions:

a) in the equations of forces the relative velocity of the center of gravity of the aircraft functionally have influence on the force of inertia, as well as the carrying speed of the center of gravity,

b) in the equations of moments of speed the changes of the moments of inertia, as, for example:

$$\left(\frac{dI_{x_1}}{dt}\right)_{mc}, \left(\frac{dI_{y_1}}{dt}\right)_{mc}, \left(\frac{dI_{z_1}}{dt}\right)_{mc}, \left(\frac{dI_{x_1 y_1}(x)}{dt}\right)_{mc}, \left(\frac{dI_{x_1 y_1}(y)}{dt}\right)_{mc},$$

have an influence on the damping qualities of the aircraft. The coefficients, characterizing their own damping of the aircraft, are $m_{x_1}^{\omega x_1}$, $m_{x_1}^{\omega y_1}$, $m_{y_1}^{\omega x_1}$, $m_{y_1}^{\omega y_1}$, $m_{z_1}^{\omega z_1}$, having negative numerical values. Consequently, so that the corresponding speeds of change of moments of inertia they improved the damping qualities of the aircraft, it is necessary so that $\left(\frac{dI_{x_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{y_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{z_1}}{dt}\right)_{mc}$ would be positive values, and a $\left(\frac{dI_{x_1 y_1}(x)}{dt}\right)_{mc}$ и $\left(\frac{dI_{x_1 y_1}(y)}{dt}\right)_{mc}$ — negative.

In connection with these conclusions it is interesting to explain, under what conditions the corresponding derivatives of moments of inertia have positive or negative values. Before being occupied with the analysis of these conditions, we will introduce formulas, determining the derivatives, and then on the basis of the formulas we will find the conditions of positiveness and negativity of these magnitudes.

2. Determination of speeds of change of the center of gravity and moments of inertia of an aircraft at constant mass.

1. Determination of speeds $\left(\frac{dx_{cg}}{dt}\right)_{mc}$, $\left(\frac{dy_{cg}}{dt}\right)_{mc}$.

In the determination of the speed of the center of gravity of an aircraft (for constant mass) we will originate from the consideration that the sum of static moments of the elementary masses of an aircraft is equal to the static moment of the entire mass:

$$\sum_{i=1}^{n} m_i dx = M dx_{cg}, \tag{18a}$$

$$\sum_{i=1}^{n} m_i dy = M dy_{cg}. \tag{18b}$$

The left-hand side of equation (18a) is presented in the following form:

$$\sum_{i=1}^{n} m_i (x_{cg} + x_1 + dx) - \sum_{i=1}^{n} m_i (x_{cg} + x_1) = \sum_{i=1}^{n} m_i dx, \tag{19}$$

where x_1 are the coordinates of masses relative to the system, passing through the center of gravity,

X_{cg} is the coordinate of the center of gravity of an aircraft relative the original system of coordinates.

The static moment of an aircraft in a moment of time t + dt relative to the original system of coordinates is equal to $\sum_{i=1}^{n} m_i (x_{cg} + x_1 + dx)$, where $\sum_{i=1}^{n} m_i$ at t + dt are equal to M_2, $M_1 = M_2 + dM$. In order to obtain the static moment of an aircraft in a moment of time t + dt relative to the original system of coordinates at constant mass, i.e. at M_1, it is necessary to add to the last expression the static moment of the separated mass from the exhaust nozzle of the engine after time dt:

$$\sum_{(M_1)} m_i (x_{cg} + x_1 + dx) = \sum_{(M_2)} m_i (x_{cg} + x_1 + dx) + dM x_n, \tag{20}$$

where x_H, y_H are the coordinates of the exhaust nozzle of the engine relative to the original axes of coordinates.

Having substituted the obtained equality in (19), we will have:

$$\sum_{(M_0)} m_i(x_{u\,T} + x_1 + dx) - \sum_{(M_0)} m_i(x_{u\,T} + x_1) + dMx_н = Mdx_{u\,T}$$

or

$$\frac{dx_{u\,T}}{dt}(m = \text{const}) = \frac{1}{M}\,\frac{d(Mx_{u\,T})}{dt} + \frac{1}{M}\left|\frac{dM}{dt}\right| x_н.$$

Having differentiated in the preceding equality the expression $\frac{d(Mx_{u\,T})}{dt}$, we obtain:

$$\frac{dx_{u\,T}}{dt}(m = \text{const}) = \frac{dx_{u\,T}}{dt}(m = \text{var}) + \frac{1}{M}\left|\frac{dM}{dt}\right| x_0 \qquad (21a)$$

and analogously

$$\frac{dy_{u\,T}}{dt}(m = \text{const}) = \frac{dy_{u\,T}}{dt}(m = \text{var}) + \frac{1}{M}\left|\frac{dM}{dt}\right| y_0, \qquad (21b)$$

where $x_0 = x_H - x_{cg}$ are the coordinates of the exhaust nozzle of the engine, $y_0 = y_H - y_{cg}$ – relative to the systems of coordinates, passing through the center of gravity.

Spreading the obtained equality on the aircraft with a turbojet engine (TRD), when besides the division of masses takes place also their connection, we obtain:

$$\frac{dx_{u\,T}}{dt}(m = \text{const}) = \frac{dx_{u\,T}}{dt}(m = \text{var}) + \frac{1}{M}\left|\frac{dM}{dt}\right| x_0(M) - \frac{1}{M}\left|\frac{dm}{dt}\right| x_0(m),$$

$$\frac{dy_{u\,T}}{dt}(m = \text{const}) = \frac{dy_{u\,T}}{dt}(m = \text{var}) + \frac{1}{M}\left|\frac{dM}{dt}\right| y_0(M) - \frac{1}{M}\left|\frac{dm}{dt}\right| y_0(m),$$

where $\frac{dM}{dt}$ – is the second expenditure of separated masses,

$\frac{dm}{dt}$ – is the second expenditure of joined masses.

If one considers that for an aircraft with a turbojet engine with an accuracy sufficient for practice it is possible to take

$$\frac{dy_{u\,T}}{dt}(m = \text{var}) = \frac{dx_{u\,T}}{dt}(m = \text{var}) \approx 0, \quad \left|\frac{dm}{dt}\right| \approx \left|\frac{dM}{dt}\right|,$$

then we find:

$$\frac{dx_{\text{ц т}}}{dt}(m = \text{const}) = \frac{1}{M}\left|\frac{dM}{dt}\right|(x_{0\,(M)} - x_{0\,(m)}),$$
$$\frac{dy_{\text{ц т}}}{dt}(m = \text{const}) = \frac{1}{M}\left|\frac{dM}{dt}\right|(y_{0\,(M)} - y_{0\,(m)}). \quad (22)$$

We obtained simple formulas for determination of the speed of the center of gravity of an aircraft at m = const.

2. <u>Determination of the speed of change of moments of inertia of an aircraft relative to the center of gravity.</u>

Let during the time dt the coordinates of masses m_1 relative to the system of coordinates, passing through the center of gravity, be changed to dx, dy. We find expressions of moments of inertia of the aircraft with respect to the axes, passing through the center of gravity at constant mass. The moment of inertia of the aircraft I_{x_1} in a moment of time t + dt (with a decreased mass on dM) relative to the former position of the center of gravity is determined by the formula:

$$I_{x_1(M_2)} = I^{*}_{x\,\text{ц т}} + M_2 dy^2_{\text{ц т}},$$

where mass M_2 corresponds to time t + dt. So that we obtain a moment of inertia of the aircraft in moment t + dt relative to the former center of gravity with mass M_1, it is necessary to value $I_{x_1}(M_2)$ to add its own moment of inertia of the discarded mass of fuel and magnitude, equal to the product of the second expenditure of mass of fuel on the square of coordinates of the output section of the engine relative to the center of gravity and on dt.

Thus, the moment of inertia I_{x_1} of mass M_1 at t + dt is determined by the following formula:

$$I_{x_1(M_1)} = I_{x\,\text{ц т}(M_1)} + M_2 dy^2_{\text{ц т}} + dMy^2_0 + dMr^2_x,$$

where r_x is the radius of gyration of a section of the discarded part of the fuel mixture.

To find increase of the moment of inertia I_{x_1} of mass M_1:

$$I_{x_1}(t+dt) - I_{x_1}(t) = I_{x_1\,\text{ц т}\,(M_2)} + M_2 dy^2_{\text{ц т}} + dM y_0^2 + dM r_x^2 - I_{x_1\,\text{ц т}}(M_1).$$

Having divided the preceding equality by dt, we obtain:

$$\frac{dI_{x_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{x_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(y_0^2 + r_x^2). \tag{23}$$

As can be seen from the formula, in order to determine the derivative of the moment of inertia in time, it is necessary from a graph $I_{xcg} = f(t)$ to determine the derivative and to add to it the product $\left|\frac{dM}{dt}\right|(y_0^2 + r_x^2)$.

Analogous formulas are easy to derive for $\left(\frac{dI_{y\,\text{ц т}}}{dt}\right)_{mc}$, $\left(\frac{dI_{z\,\text{ц т}}}{dt}\right)_{mc}$:

$$\frac{dI_{y_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{y_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(x_0^2 + r_y^2),$$

$$\frac{dI_{z_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{z_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|[x_0^2 + y_0^2 + r_z^2], \tag{24}$$

$$x_0 = x_{\text{н}} - x_{\text{ц т}},\quad y_0 = y_{\text{н}} - y_{\text{ц т}},$$

where x_H, y_H are the coordinates of the exhaust nozzle of the engine relative to the original axes of coordinates.

For aircraft, supplied with turbo-jet engines (TRD), in a case, when takes place division of combustible gases and connection of air masses, the preceding equalities take the form:

$$\frac{dI_{x_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{x_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(y_0^2 + r_x^2)_{(M)} - \left|\frac{dm}{dt}\right|(y_0^2 + r_x^2)_{(m)},$$

$$\frac{dI_{y_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{y_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(x_0^2 + r_y^2)_{(M)} - \left|\frac{dm}{dt}\right|(x_0^2 + r_y^2)_{(m)},$$

$$\frac{dI_{z_1\,\text{ц т}}}{dt}(m = \text{const}) = \left(\frac{dI_{z_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(x_0^2 + y_0^2 + r_z^2)_{(M)} - \left|\frac{dm}{dt}\right|(x_0^2 + y_0^2 + r_z^2)_{(m)},$$

where $\frac{dM}{dt}$ – is the second expenditure of separated mass,

$\frac{dm}{dt}$ – is the second expenditure of joined mass.

If one considers that for an aircraft with a turbojet engine exist the dependences

$$\left(\frac{dI_{x\,\text{ц т}}}{dt}\right)_{mV} \approx \left(\frac{dI_{y\,\text{ц т}}}{dt}\right)_{mV} \approx \left(\frac{dI_{z\,\text{ц т}}}{dt}\right)_{mV} = 0, \ \left|\frac{dm}{dt}\right| \approx \left|\frac{dM}{dt}\right|,$$

then the preceding equalities can be presented as:

$$\begin{aligned}
\frac{dI_{x\,\text{ц т}}}{dt}(m = \text{const}) &= \left|\frac{dM}{dt}\right| [(y_0^2 + r_x^2)_{(M)} - (y_0^2 + r_x^2)_{(m)}], \\
\frac{dI_{y\,\text{ц т}}}{dt}(m = \text{const}) &= \left|\frac{dM}{dt}\right| [(x_0^2 + r_y^2)_{(M)} - (x_0^2 + r_y^2)_{(m)}], \\
\frac{dI_{z\,\text{ц т}}}{dt}(m = \text{const}) &= \left|\frac{dM}{dt}\right| [(x_0^2 + y_0^2 + r_z^2)_{(M)} - (x_0^2 + y_0^2 + r_z^2)_{(m)}].
\end{aligned} \qquad (25)$$

The obtained formulas are very convenient to use for determination of the speeds of moments of inertia of an aircraft at constant mass, necessary for calculation of the dynamics of the aircraft.

3. <u>Determination of</u> $\left(\frac{dI_{x_1y_1}(x)}{dt}\right)_{mc}$, $\left(\frac{dI_{x_1y_1}(y)}{dt}\right)_{mc}$.

For determination of derivatives of the product of inertia in time at a separate change of coordinates it is necessary to directly calculate in the function of time any of the two partial derivatives. Determining, for instance $\left(\frac{dI_{x_1y_1}(x)}{dt}\right)_{mc} = \int\limits_{(m)} dm \frac{dx}{dt} y$, the second partial derivative, we obtain by means of subtraction from the total derivative $\left(\frac{dI_{x_1y_1}(x)}{dt}\right)_{mc}$.

We will show a method of determination of the total derivative of the product of inertia at constant mass. In the moment of time t + dt the product of inertia is equal to $I_{x_1y_1}(M_2)$, and relative to the system of coordinates, passing through the center of gravity in

the moment of time t, it equals $I_{x_1y_1}(M_2) + M_2 dx_{cg} \cdot dy_{cg}$.

The product of inertia in moment of time $t + dt$ from mass M_1 equals the sum of the product of inertia from mass M_2 and the product of the separated mass after time dt on coordinates x_0, y_0 of the exhaust nozzle

$$I_{x_1y_1}(M_1) = I_{x_1y_1}(M_2) + M_2 dx_{\text{ц т}}\, dy_{\text{ц т}} + dMx_0\, y_0.$$

We will record the increase of the moment of inertia after time dt:

$$dI_{x_1y_1}(m = \text{const}) = I_{x_1y_1}(M_2) - I_{x_1y_1}(M_1) + M_2 dx_{\text{ц т}}\, dy_{\text{ц т}} + dMx_0 y_0.$$

Dividing the increase by dt, we have:

$$\frac{dI_{x_1y_1}}{dt}(m = \text{const}) = \left(\frac{dI_{x_1y_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right| x_0 y_0. \qquad (26)$$

For an aircraft with a turbojet engine the preceding equality takes the form:

$$\frac{dI_{x_1y_1}}{dt}(m = \text{const}) = \left(\frac{dI_{x_1y_1\,\text{ц т}}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right| x_0\, y_0\,(M) - \left|\frac{dm}{dt}\right| x_0\, y_0\,(m).$$

If one considers that $\left(\frac{dI_{x_1y_1}}{dt}\right)_{mV} \approx 0,\ \left|\frac{dm}{dt}\right| \sim \left|\frac{dM}{dt}\right|$, we obtain:

$$\frac{dI_{x_1y_1}}{dt}(m = \text{const}) = \left|\frac{dM}{dt}\right| [x_0\, y_0\,(M) - x_0\, y_0\,(M)].$$

Thus, in order to determine the total derivative from the product of inertia in time at m = const, it is necessary to determine from a graph of the product of inertia relative to the system of coordinates, passing through the center of gravity, the time derivative and to add the product $\left|\frac{dM}{dt}\right| x_0\, y_0$.

3. Evaluation of the influence of variability of mass in the motion equations of an aircraft.

Equations of motion of an aircraft of variable mass are

distinguished from analogous equations of motion of a solid aircraft into every moment of time by additional members, considering the speed of change of the center of gravity and the speed of change of the moments of inertia of the aircraft at constant mass. Above, on the basis of analysis of equations of motion of an aircraft (16), we arrived to the conclusion that for improvement of damping it is necessary, in order to $\left(\frac{dI_{x_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{y_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{z_1}}{dt}\right)_{mc}$ would be positive values.

Now we will define the qualitative conditions, with which these requirements are executed. For this purpose we will turn to the formulas, determining the speed of change of the moments of inertia at constant mass.

Let us consider, for example, the speed of change of the moment of inertia around axis y_1, determined by the formula:

$$\left(\frac{dI_{y_1}}{dt}\right)_{mc} = \left(\frac{dI_{y_1}}{dt}\right)_{mV} + \left|\frac{dM}{dt}\right|(x_0^2 + r_y^2).$$

As can be seen from the last formula so that $\left(\frac{dI_{y_1}}{dt}\right)_{mc}$ is greater than zero, fulfillment of the following condition is necessary:

$$\left|\frac{dM}{dt}\right|(x_0^2 + r_{y_1}^2) > \left|\frac{dI_{y_1}}{dt}\right|_{mV}.$$

Fulfillment of this condition is possible in two versions.

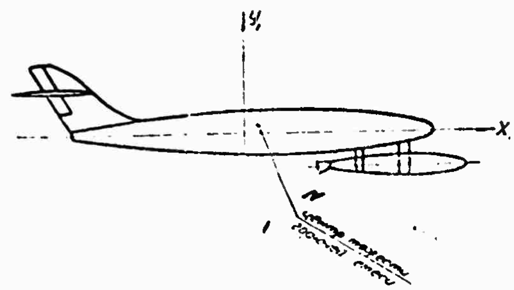

Fig. 5. Diagram of the location of the engine in front of the center of gravity of an aircraft. 1) Fuel mixture; 2) center of gravity.

First version: the exhaust nozzle of the engine is located in front of the center of gravity of the aircraft, and the center of gravity of the fuel mixture is located relative to the center of gravity of the aircraft at significantly smaller distance (Fig. 5).

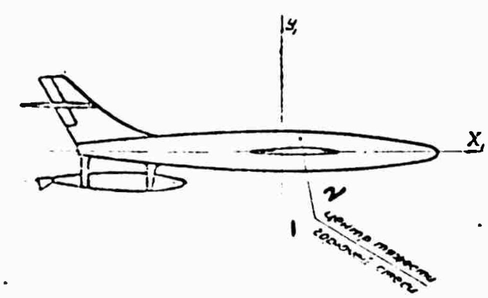

Fig. 6. Diagram of the location of the engine behind the center of gravity of an aircraft. 1) Fuel mixture; 2) center of gravity.

Second verson; the exhaust nozzle of the engine is located behind the center of gravity of the aircraft at a large distance, and the center of gravity of the fuel mixture relative to the center

of gravity of the aircraft at a large distance, and the center of gravity of the fuel mixture relative to the center of gravity of the aircraft is located at a significantly smaller distance (Fig. 6).

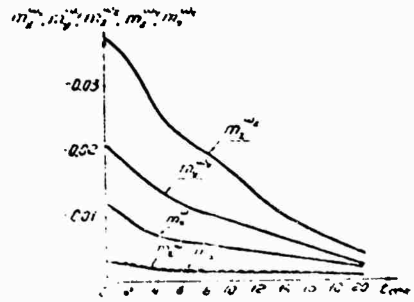

Fig. 7.

In the arrangement of an aircraft one whould consider these considerations. We will note however, so that $\left(\frac{dI_{y_1}}{dt}\right)_{mc} > 0$, is actually a realizable arrangement according to the second verson. Analogous reasonings are spread also on the speed of change of the moments of inertia around axes x_1, z_1. The above-introduced analysis has a qualitative character. However in practice it can appear that the calculation of additional members numerically have little influence on equations of motion of aircraft. Therefore it is expedient to produce a numerical evaluation of additional members for a judgement about the boundaries of application of these equations. Let us consider an aircraft, accomplishing takeoff with the help of a liquid-fuel rocket engine from the method of horizontal flight. Aerodynamic and flight characteristics of an aircraft during take off, as, for example, $m_{x_1}^{\omega_{x_1}}$, $m_{y_1}^{\omega_{y_1}}$, $m_{z_1}^{\omega_{z_1}}$. $m_{x_1}^{\omega_{y_1}}$, $m_{y_1}^{\omega_{x_1}}$, Sql, Sqb_a, are shown in graphs (Fig. 7).

The inertial weight characteristics of an aircraft during take

off in a function of time are given on the graphs (Fig. 8).

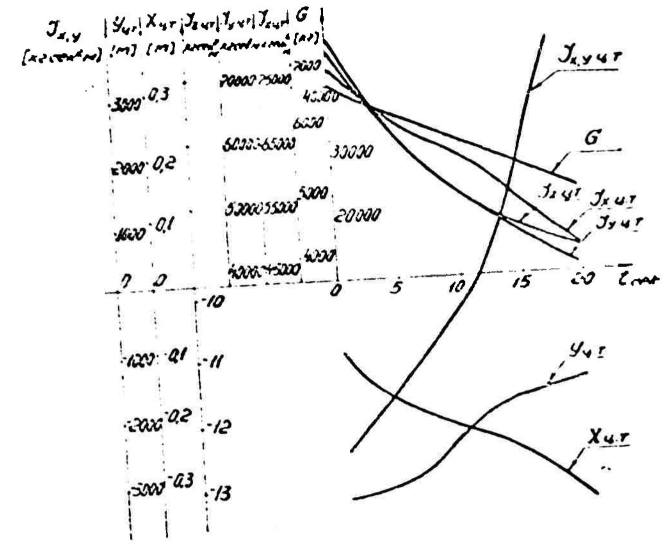

Fig. 8.

Differentiating in time the curves of change G, I_{x_1}, I_{y_1}, I_{z_1}, x_{cg}, y_{cg}, we will find the speed of change of these magnitudes at variable mass, and namely:

$$\left(\frac{dx_{c.g.}}{dt}\right)_{mV}, \left(\frac{dy_{c.g.}}{dt}\right)_{mV}, \left(\frac{dI_{x_1}}{dt}\right)_{mV}, \left(\frac{dI_{y_1}}{dt}\right)_{mV}, \left(\frac{dI_{z_1}}{dt}\right)_{mV}, \frac{dM}{dt}, \left(\frac{dI_{x_1y_1}}{dt}\right)_{mV}.$$

Using then formulas (21), (23), (26), we determine the speeds of change of the center of gravity and moments of inertia of an aircraft at constant mass:

$$\left(\frac{dx_{c.g.}}{dt}\right)_{mc}, \left(\frac{dy_{c.g.}}{dt}\right)_{mc}, \left(\frac{dI_{x_1}}{dt}\right)_{mc}, \left(\frac{dI_{y_1}}{dt}\right)_{mc}, \left(\frac{dI_{z_1}}{dt}\right)_{mc}, \left(\frac{dI_{x_1y_1}}{dt}\right)_{mc}.$$

On the basis of data of the numerical calculation of the above-considered aircraft we will produce calculation of coefficients of additional damping from the variability of mass of the aircraft.

From equations of motion of an aircraft (16) it follows that coefficients of additional damping of an aircraft from variability of mass are determined by the following formulas:

$$\Delta m_{x_1}^{\omega_{x_1}} = \left(\frac{dI_{x_1}}{dt}\right)_{mc} \frac{1}{Sql}, \quad \Delta m_{y_1}^{\omega_{y_1}} = \left(\frac{dI_{y_1}}{dt}\right)_{mc} \frac{1}{Sql},$$

$$\Delta m_{z_1}^{\omega_{z_1}} = \left(\frac{dI_{z_1}}{dt}\right)_{mc} \frac{1}{Sqb_a}, \quad \Delta m_{x_1}^{\omega_{y_1}} = 2\left(\frac{dI_{x_1y_1}(x)}{dt}\right)_{mc} \frac{1}{Sql},$$

$$\Delta m_{y_1}^{\omega_{x_1}} = 2\left(\frac{dI_{x_1y_1}(y)}{dt}\right)_{mc} \frac{1}{Sql}.$$

Results of the calculations of these coefficients are given in the table.

t сек	$\Delta m_{y_1}^{\omega_{y_1}}$	$\Delta m_{z_1}^{\omega_{z_1}}$	$\Delta m_{x_1}^{\omega_{y_1}}$	$\Delta m_{y_1}^{\omega_{x_1}} \cdot 10^5$	$\frac{\Delta m_{y_1}^{\omega_{y_1}}}{m_{y_1}^{\omega_{y_1}}}$ %	$\frac{\Delta m_{z_1}^{\omega_{z_1}}}{m_{z_1}^{\omega_{z_1}}}$ %	$\frac{\Delta m_{x_1}^{\omega_{y_1}}}{m_{x_1}^{\omega_{y_1}}}$ %	$\frac{\Delta m_{y_1}^{\omega_{x_1}}}{m_{y_1}^{\omega_{x_1}}}$ %
0	0,024	0,031	0,002	31	110,8	82,5	−64,9	−10,7
2	0,0036	0,012	0,0097	7,8	51,5	35	−28,6	−3,25
4	0,0062	0,0085	[illegible]	9,2	44,07	32,17	−25,86	−5,124
6	0,0053	0,007	0,0004	11	45,26	33,11	−24,0	−6,39
8	0,0055	0,0076	0,0004	16	54,2	39,2	−26,6	−11,5
10	0,007	0,0099	0,0005	10,2	79,3	61,62	−36,9	−8,5
12	0,0088	0,012	[illegible]	8,2	118,4	95,3	−53,1	−8,2
14	0,012	0,017	[illegible]	5,5	201	176	−71,9	−6,2
16	0,0089	0,013	[illegible]	2,4	167,8	174	−57,6	−2,99
18	0,0054	0,0076	[illegible]	0,7	175	121,9	−38,4	−1,2

In comparing the numerical values of additional coefficients of damping with its own coefficients of damping, we notice that in the equations of moments (16) the additional members

$\left(\frac{dI_{x_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{y_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{z_1}}{dt}\right)_{mc}$, $\left(\frac{dI_{x_1 y_1}(x)}{dt}\right)_{mc}$ can considerably have an influence on the damping qualities of the aircraft and it is impossible to disregard them. In projection of an aircraft one should consider this circumstance. We will note however, that in the equations of motion of an aircraft one whould consider the coefficients of additional damping, when they exceed 10% in reference to its own damping of the aircraft, since the aerodynamic coefficients are usually given with an accuracy to 10%.

In the equations of forces (16) it is possible to disregard the members, considering the speed of change of the center of gravity at constant mass $\left(\frac{dx_{ц т}}{dt}\right)_{mc}$, $\left(\frac{dy_{ц т}}{dt}\right)_{mc}$, in view of their insignificant influence.

In conclusion we will consider the case of flight of an aircraft, when its mass changes abruptly, for example due to booster separation. Do there appear in this case additional damping coefficients in the equations of motion? We will show that the speeds of change of moments of inertia at constant mass in this case equals zero. Let us consider, for example, the speed of change of the moment of inertia of an aircraft with respect to axis y_1 – $\left(\frac{dI_{y_1}}{dt}\right)_{mc}$ at booster separation.

Let its own moment of inertia of the booster equal $m_{yc} r_y^2$, where m_{yc} is the mass of the booster,

r_y is the radius of gyration.

Then the speed of change of the moment of inertia of the aircraft relative to the center of gravity of the aricraft at variable mass will be:

$$\left(\frac{dI_{y_1}}{dt}\right)_{m.v} = -\frac{m_{yc}(x_0^2 + r_y^2)}{\Delta t} (\Delta t \to 0).$$

The speed of change of the moment of inertia of the aircraft at constant mass is determined, as it is known, by the following dependence:

$$\left(\frac{dI_{y_1}}{dt}\right)_{mc} = \left(\frac{dI_{z_1}}{dt}\right)_{mV} + m_{yc}\frac{(x_0^2 + r_y^2)}{\Delta t}\,(\Delta t \to 0).$$

Substituting the preceding in the obtained equality, we will have:

$$\left(\frac{dI_{y_1}}{dt}\right)_{mc} = \frac{-m_{yc}(x_0^2 + r_y^2) + m_{yc}(x_0^2 + r_y^2)}{\Delta t}\,(\Delta t \to 0) = 0.$$

Consequently, we showed that the speed of change of the moments of inertia in this case is equal to zero and therefore in the equations of motion members of additional damping are absent. However, the value of mass should be new and the aerodynamic coefficients calculated correspondingly to the new position of the center of gravity. Equations of motion of aircraft are solved under the original conditions, corresponding to the moment of booster separation.

Conclusions

On the basis of what has been stated it follows that the motion of an aircraft with a jet engine, the mass and moments of inertia of which greatly vary, should be depicted with equations (16), considering the variability of mass. If the mass is variable and with this takes place the relative motion of the fuel mixture in a connected system of coordinates of the aircraft, then in the equations of motion of the aircraft is considered the variability of the mass and the additional members, considering the damping influence on aircraft of relative motion of fuel mixture.

It is very important in the projection of an aircraft to correctly arrange the position of the engine (booster) and tanks of fuel mixture relative to the center of gravity of the aircraft. For

an increase of damping of an aircraft it is expedient to ~~dispose~~ place the jet engine behind the center of gravity at a significant distance. Additional damping from variability of mass one should consider in a case, when the coefficients $\left(\frac{dI_{x_1}}{dt}\right)_{mc}\frac{1}{Sql}\frac{100}{m_{x_1}^{\omega_{x_1}}}$, $\left(\frac{dI_{y_1}}{dt}\right)_{mc}\frac{1}{Sql}\frac{100}{m_{y_1}^{\omega_{y_1}}}$ etc. are greater than 10%.

In the equations of forces (16) it is possible to disregard the members, considering the relative motion of the center of gravity of an aircraft:

$$\left(\frac{dx_{\text{ц т}}}{dt}\right)_{mc},\ \left(\frac{dy_{\text{ц т}}}{dt}\right)_{mc}.$$

If, however, the mass of an aircraft is variable, but with this there is no relative motion of separated masses in the connected system of coordinates (booster separation), then the members of equations (16), considering the additional damping, equal zero. In this case equations (16) consider only the absolute change of mass, the moments of inertia and the change of the center of gravity of the aircraft.

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